from 1 ft^2 of surface area centered at any point along the isodose line.

The scattered radiation characteristics of several possible SNAP 8 radiator configurations have been analyzed. Table 3 presents the results of one such study. In general, the fast neutron dose from unshielded radiators exceeds 10^{12} nvt for reactor-to-payload distances of less than about 40 ft.

One way to reduce the fast neutron dose scattered from the radiator to the payload is to shield the radiator. For most radiator configurations except a conical radiator, however, shielding the radiator requires increasing the solid angle shadowed by the shadow shield, thus increasing the weight of the shield. It should be noted that, in general, the portion of shield which shadows the radiator must be nearly as thick as the payload shadow shield. Otherwise, the neutrons emerging from the surface of the thinner shield constitute a more powerful surface source than the outer surface of the payload shadow shield. Thus, a shield that reduces the radiation incident upon the radiator can be prohibitively heavy if the radiator extends out like wings, for example.

On this basis, it is recommended that the conical radiator concept be given serious consideration for the SNAP 8 spacecraft. This concept has the inherent disadvantage that the radiating area is restricted to the external surface only but retains the extremely desirable characteristic that all spacecraft components and structures can be located behind a relatively small shadow shield.

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Prediction of Interstage Pressure in Multistage Solid-Propellant Rocket Systems

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A mathematical model has been developed for accurately predicting the interstage pressure-time transient that exists during the separation of two stages of a solid-propellant rocket system. The system of equations, composed of mass, energy, and force balances for the rocket chamber and interstage region, is solved simultaneously to yield predictions of the time dependence of chamber temperature and pressure together with interstage pressure, temperature, and volume. Sample calculations and prediction-results for a typical multistage rocket system are included.

Nomenclature

a = reaction-time constant in Summerfield equation
a_l = acceleration of lower-stage rocket relative to earth, ft/sec²
a_r = acceleration of upper-stage rocket relative to lower stage, ft/sec²
a_u = acceleration of upper-stage rocket relative to earth, ft/sec²
A_b = propellant burning area, ft²
A_e = combined cross-sectional area of nozzles at exit planes, ft²
A_g = variable cylindrical surface-area of interstage gap, ft²
A_i = cross-sectional area of interstage at gap edge, ft²
A_t = combined cross-sectional area of nozzles at throats, ft²
b = diffusion-time constant in Summerfield equation
B = angle of inclination of rocket to earth at time of separation, deg
C = thrust coefficient efficiency; $0.93 \leq C \leq 0.99$
C_d = interstage gap discharge coefficient
C_f = thrust coefficient of upper stage
C_{p_c} = heat capacity of chamber gases, Btu/lb-°R
C_{p_i} = heat capacity of interstage gases at constant pressure, Btu/lb-°R
C_{p_0} = heat capacity of igniter gases, Btu/lb-°R
C_{p_s} = heat capacity of interstage surfaces, Btu/lb-°R
C_{v_c} = heat capacity of chamber gases at constant volume, Btu/lb-°R
C_{v_i} = heat capacity of interstage gases at constant volume, Btu/lb-°R
C_w = rocket nozzle mass flow coefficient, sec⁻¹
C_{w_g} = interstage gap mass flow coefficient, sec⁻¹
d_b = propellant density, lb/ft³
F_r = residual thrust of lower stage motor, lbf
g = acceleration of gravity, ft/sec²
g_c = gravitational constant, lb-ft/lbf-sec²
h_i = heat transfer coefficient of interstage gases to interstage surfaces, Btu/hr-ft²-°F
k = ratio of specific heats, C_p/C_v, for exhaust
K = fraction of exhaust internal-energy transferred to interstage walls as heat loss (if heat loss equals 80%, K equals 0.80)
L = thrust coefficient correction for divergence angle
m_0 = mass flow-rate of igniter, lb/sec
M_e = effective molecular weight of chamber-gas/condensed-phase mixture, lb/lb-mole
M_l = mass of lower stage, lb
M_u = mass of upper stage, lb
P_c = chamber pressure, lbf/ft²
P_i = interstage pressure, lbf/ft²
P_e = pressure at nozzle exit plane, lbf/ft²
q_i = rate of heat loss from interstage gases to interstage walls, Btu/sec
Q_i = total heat loss from interstage gases to interstage walls, Btu
q_c = heat loss from chamber gases to chamber walls, Btu/sec
r = propellant burning rate, fps
r_i = radius of interstage at gap edge, ft
R = universal gas constant, ft-lbf/lb-mole-°R
S = total surface area of interstage walls, ft²

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T_c = chamber temperature, °R
T_i = interstage temperature, °R
T_0 = igniter gas temperature, °R
T_r = reference temperature, °R
T_s = surface temperature of interstage walls, °R
ΔU_c = energy of combustion of propellant at reference temperature, Btu/lb
V_c = volume of chamber, ft^3
V_i = volume of interstage including gap, ft^3
V_{i0} = initial volume of interstage, ft^3
W_s = total mass of interstage walls, lb
x = width of interstage gap, ft
ϵ = nozzle cone expansion ratio
θ = time from ignition, sec

An improved mathematical model accurately predicts the interstage pressure-time transient that exists during the separation of two stages of a solid-propellant rocket system (see Fig. 1). The prediction is obtained through simultaneous solution of mass, energy, and force balances for the rocket chamber and interstage region. Solution of the balances yields predictions of the time dependence of upper-stage chamber temperature and pressure together with interstage pressure, temperature, and volume. The prediction of these values extends from the time of upper-stage ignition to the time when the interstage pressure begins to decrease.

Other methods have been previously suggested for predicting interstage pressure. Information from the preliminary unpublished papers of Dekker[1] and Johnson[2] contributed substantially to the current mathematical model.

The model described in this article obtains greater accuracy and applicability than earlier models. The accuracy is improved through consideration of a nonisothermal condition for the rocket chamber and interstage region. Greater applicability is attained by eliminating the necessity of furnishing upper-stage rocket-chamber pressure transients. The mathematical model is applicable to a variety of multistage, solid-propellant rocket systems and has been validated through comparison to experimentally measured interstage pressure.

A. Assumptions and Limitations

The algebraic development of the mathematical model is based on the rocket model shown in Fig. 1 with the following assumptions and limitations:

1) The thermodynamic system for the upper-stage rocket chamber includes the void chamber volume and the propellant burning-surface. The igniter is treated as part of the physical surroundings.

2) The propellant surface is assumed to ignite uniformly and instantaneously at time zero. Without a consideration of the kinetics of combustion involved, it is difficult to predict how the surface is actually ignited. However, if experimental chamber-pressure transients are combined with the mathematical model, a time delay is automatically included in the interstage pressure prediction.

3) The propellant burning-surface area is assumed constant during stage separation.

4) The temperature of the propellant burning-surface is assumed constant.

5) The heat transfer from hot gases to the propellant subsurface is assumed negligible, since the propellant surface regression rate exceeds the rate of subsurface heat conduction.

6) The chamber volume is assumed constant during the period of stage separation.

7) The gaseous temperature and pressure are assumed uniform throughout the rocket chamber at any specific time.

8) The rocket exhaust gases are assumed to obey the perfect gas laws.

9) The kinetic energy of the chamber gases is assumed negligible compared to the thermal effects present.

10) The heat capacities of the gases are assumed constant. Actually, heat capacity varies only 30% over the entire temperature range of interest.

11) The heat loss to interstage walls is assumed to be a constant fraction K of the thermal energy of the gaseous products entering the interstage region. In reality, g_i is a function of time, since the temperatures of the gases and interstage walls change with time. A second method is presented which considers the time dependence of the heat loss to the interstage region. However, this new approach has not yet been evaluated.

12) The combustion products are assumed to be in frozen equilibrium as they pass through the nozzle and into the interstage region. This fixes the molecular weight as a constant.

13) The combustion products in the rocket nozzle and the interstage gap are assumed to be in choke flow. For an isentropic system, choke flow occurs between 1 and 2 msec, when the gaseous pressures on opposite sides of the particular aperture are in the ratio 2:1.

14) The nozzle discharge and thrust coefficients used are based on isentropic thermodynamic relationships.

15) The upper-stage exhaust of the multistage rocket system is assumed to exert its full force on the lower stage throughout the staging period.

16) The forces exerted on the multistage rocket system from aerodynamic drag are assumed to be zero during the staging period.

B. Derivation of Mathematical Model

These assumptions enable the following material, energy, and force balances to be written for a multistage, solid-propellant rocket system. For simplicity, a two-stage rocket is treated in this article.

This development is necessarily brief. A more comprehensive discussion is presented in the classified literature.[3]

1. Material Balance for Rocket Chamber

The general mass balance for a rocket chamber during ignition is

$$\overset{a)}{\begin{bmatrix}\text{rate of mass into cham-}\\\text{ber from igniter charge,}\\\text{lb/sec}\end{bmatrix}} + \overset{b)}{\begin{bmatrix}\text{mass production-rate at}\\\text{propellant burning-sur-}\\\text{face, lb/sec}\end{bmatrix}} = \overset{c)}{\begin{bmatrix}\text{mass flow-rate through}\\\text{rocket nozzle, lb/sec}\end{bmatrix}} + \overset{d)}{\begin{bmatrix}\text{mass accumulation-rate}\\\text{in rocket chamber, lb/}\\\text{sec}\end{bmatrix}} \quad (1)$$

For variable chamber temperature and pressure, the algebraic expression for each term is

$$\overset{a)}{m_0} + \overset{b)}{A_b d_b r} = \overset{c)}{C_w P_c A_t} + \overset{d)}{\frac{M_e V_c}{R}\frac{d}{d\theta}\left(\frac{P_c}{T_c}\right)} \quad (2)$$

2. Energy Balance for Rocket Chamber

$$\overset{a)}{\begin{bmatrix}\text{rate of energy into cham-}\\\text{ber from igniter, Btu/}\\\text{sec}\end{bmatrix}} = \overset{b)}{\begin{bmatrix}\text{rate of heat transfer}\\\text{from system to sur-}\\\text{roundings, Btu/sec}\end{bmatrix}} + \overset{c)}{\begin{bmatrix}\text{rate of energy expelled}\\\text{through rocket nozzle,}\\\text{Btu/sec}\end{bmatrix}} + \overset{d)}{\begin{bmatrix}\text{rate of accumulation of}\\\text{energy in rocket cham-}\\\text{ber, Btu/sec}\end{bmatrix}} \quad (3)$$

Expressed analytically, Eq. (3) becomes

$$\overset{a)}{m_0C_{r0}(T_0 - T_r)} = \overset{b)}{q_c} + \overset{c)}{C_wA_tP_cC_{pc}(T_c - T_r)} + \overset{b)}{\Delta U_cA_bd_br} + \frac{C_{vc}M_cV_c}{RT_c}\left[(T_c - T_r)\frac{dP_c}{d\theta} + \frac{P_cT_r}{T_c}\frac{dT_c}{d\theta}\right] \quad (4)$$

3. Mass Balance for Interstage Region

a) [rate of mass into interstage from rocket, lb/sec] = b) [rate of mass elected through interstage separation gap, lb/sec] + c) [rate of accumulation of mass in interstage region, lb/sec] (5)

By writing the terms mathematically for variable interstage temperature, pressure, and volume, Eq. (5) becomes

$$\overset{a)}{C_wP_cA_t} = \overset{b)}{C_{wg}P_iA_g} + \overset{c)}{\frac{M_e}{R}\frac{d}{d\theta}\left(\frac{P_iV_i}{T_i}\right)} \quad (6)$$

4. Energy Balance for Interstage Region

a) [rate of energy into interstage from rocket, Btu/sec] = b) [rate of heat transfer from system to surroundings, Btu/sec] + (7)

c) [rate of energy ejected through interstage gap, Btu/sec] + d) [rate of accumulation of energy in interstage region, Btu/sec]

or, mathematically,

$$\overset{a)}{C_wP_cA_tC_{pc}(T_c - T_r)} = \overset{b)}{q_i} + \overset{c)}{C_{wg}P_iA_gC_{pi}(T_i - T_r)} + \overset{d)}{\frac{C_{vi}M_e}{R}\left[\frac{d}{d\theta}(P_iV_i) - \frac{d}{d\theta}\left(\frac{P_iV_iT_r}{T_i}\right)\right]} \quad (8)$$

5. Force Balances for Two-Stage Rocket System[1]

The forces acting on the system are illustrated in Fig. 1.

a. Force balance for upper stage

The general equation for the upper-stage force balance is

a) [rocket motor thrust, lbf] + b) [force from interstage pressure, lbf] + c) [force from axial component of gravity, lbf] + (9)

d) [force from aerodynamic drag, lbf] = e) [product of mass and acceleration of upper stage, lbf]

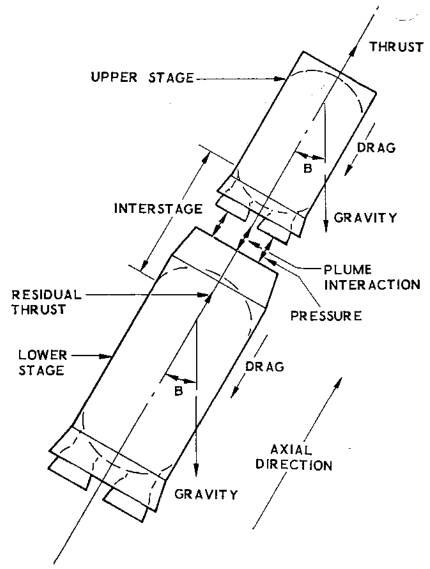

Fig. 1 Forces affecting a two-stage rocket system at separation.

Neglecting aerodynamic drag, the algebraic terms of the general equation are

$$\overset{a)}{C_fP_cA_t} + \overset{b)}{P_i(A_i - A_e)} - \overset{c)}{\left(\frac{M_ug}{g_c}\right)\cos B} = \overset{e)}{\frac{M_ua_u}{g_c}} \quad (10)$$

b. Force balance for lower stage

a) [residual thrust of lower stage, lbf] + b) [force on lower stage from upper-stage exhaust, lbf] + c) [force from interstage pressure, lbf] + d) [force from axial component of gravity, lbf] + (11)

e) [force from aerodynamic drag, lbf] = f) [product of mass and acceleration of lower stage, lbf]

or, neglecting drag,

$$\overset{a)}{F_r} - \overset{b)}{C_fP_cA_t} - \overset{c)}{P_i(A_i - A_e)} - \overset{d)}{(gM_l/g_c)\cos B} = \overset{f)}{(M_la_l/g_c)} \quad (12)$$

c. Combined force balance for two-stage rocket system

The relative acceleration of the upper stage from the lower stage is

$$a_r = a_u - a_l \quad (13)$$

By geometry, the gap area A_g can be related to the length of gap separation x and the interstage volume V_i by

$$A_g = 2\pi r_ix = 2\pi r_i[(V_i - V_{i0})/A_i] \quad (14)$$

Since

$$V_i = A_ix + V_{i0} \quad (15)$$

then

$$d^2V_i/d\theta^2 = A_i(d^2x/d\theta^2) \quad (16)$$

Table 1 Ballistic parameters for a typical two-stage rocket motor system

Item	Numerical value	Item	Numerical value
a	4×10^5	k	1.18
A_b	20 ft²	K	0.75
A_e	5 ft²	L	0.96
A_i	10 ft²	m_0	15 lb/sec
A_t	0.25 ft²	M_e	30 lb/lb-mole
b	500	M_l	2000 lb
C	0.95	M_u	6000 lb
C_{Pc}	0.43 Btu/lb-°R	R	1545 ft-lbf
C_{Pi}	0.41 Btu/lb-°R	r_i	1.8 ft
C_{P0}	0.28 Btu/lb-°R	T_0	6800°R
C_{vc}	0.36 Btu/lb-°R	T_r	537°R
C_{vi}	0.34 Btu/lb-°R	ΔU_c	−3000 Btu/lb
d_b	100 lb/ft³	V_c	1.5 ft³
F_t	2000 lbf	V_{i0}	30 ft³
g_c	32.2 lb-ft/lbf-sec²	ϵ	20

but

$$d^2x/d\theta^2 = a_r \tag{17}$$

Consequently,

$$d^2V_i/d\theta^2 = A_i a_r \tag{18}$$

Combining Eqs. (10, 12, 13, and 18) gives the basic force balance:

$$\frac{d^2V_i}{d\theta^2} = A_i g_c \left[C_f P_c A_t \left(\frac{1}{M_u} + \frac{1}{M_l}\right) + P_i(A_i - A_e)\left(\frac{1}{M_u} + \frac{1}{M_l}\right) - \frac{F_r}{M_l}\right] \tag{19}$$

6. Thrust and Mass Flow Coefficients

Equations (20) and (21) are standard equations[4] describing C_w and C_f as functions of the dependent variables, temperature and pressure. Equation (20) may be used for chamber or interstage:

$$C_w = C_d \left[\frac{M_e k g}{TR}\left(\frac{2}{k+1}\right)^{(k+1)/(k-1)}\right]^{1/2} \tag{20}$$

$$C_f = CL\left\{\frac{2k^2}{k-1}\left(\frac{2}{k+1}\right)^{(k+1)/(k-1)} \times \left[1 - \left(\frac{P_e}{P_c}\right)^{(k-1)/k}\right]\right\}^{1/2} + \epsilon\,\frac{P_e - P_i}{P_c} \tag{21}$$

where

$$\epsilon = \frac{[2/(k+1)]^{[(k+1)/2(k-1)]}}{(P_e/P_c)^{1/k}\{[2/(k-1)]\,[1-(P_e/P_c)]^{(k-1)/k}\}^{1/2}}$$

7. Burning Rate

The effect of rocket chamber pressure on the burning rate of solid composite propellants can be predicted for a wide range of pressures using the Summerfield equation[5]:

$$1/r = (a/P_c) + (b/P_c^{1/3}) \tag{22}$$

C. Solution of Mathematical Model

The complete mathematical model consists exclusively of Eqs. (2, 4, 6, 8, and 19), associated with Eqs. (14 and 20–22). The parameters r, C_w, C_f, and A_g may be eliminated from the basic mathematical model by substituting Eqs. (14 and 20–22) into Eqs. (2, 4, 6, 8, and 19). The resulting set of equations describes P_c, T_c, P_i, T_i, and V_i as functions of time θ.

These model equations may be adapted to many rocket systems by substituting the applicable rocket system ballistic parameters into the model equation and by evaluating q_i, the unsteady-state heat loss, to the interstage walls.

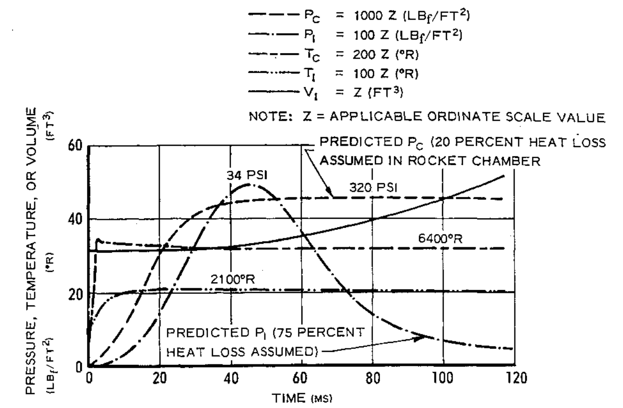

Fig. 2 Prediction of time dependency of interstage pressure for a typical two-stage rocket system: nonadiabatic case.

The evaluation of q_i by direct means is a complex problem resulting from unusual geometry and widely varying temperatures. The two approaches presented in this article consist of a simplified method developed in this section and a more rigorous treatment derived in Sec. E.

When assumption 11 is used, q_i can be eliminated from Eq. (8) by multiplying the left side of the equation by $(1 - K)$. The estimation of K requires at least one experimental evaluation of the interstage pressure transient for the rocket system of interest. To estimate K, this transient is combined with the mathematical model and the ballistic parameters for this particular system. The resulting value of K is then assumed to remain constant as the effects of other parameters (such as nozzle throat area) on the interstage pressure are determined from the mathematical model. When this method was applied to a particular rocket system, the value of K was determined to be 0.75, indicating a 75% loss of heat from the rocket exhaust gases to the interstage walls during staging. Heat loss in the rocket chamber, q_c, can in many cases be neglected, since the propellant burning-surface is treated as part of the thermodynamic system.

With these approximations for K and q_c, all terms in Eqs. (2, 4, 6, 8, and 19) are constant except P_c, T_c, V_i, T_i, P_i, and θ. These time-dependent values may now be obtained from the solution of the indicated equations with a digital or analog computer.

D. Specific Application of Mathematical Model

Using the proposed mathematical model, time-dependent predictions of chamber temperature and pressure together with interstage temperature, pressure, and volume have been made for a typical two-stage rocket system. The

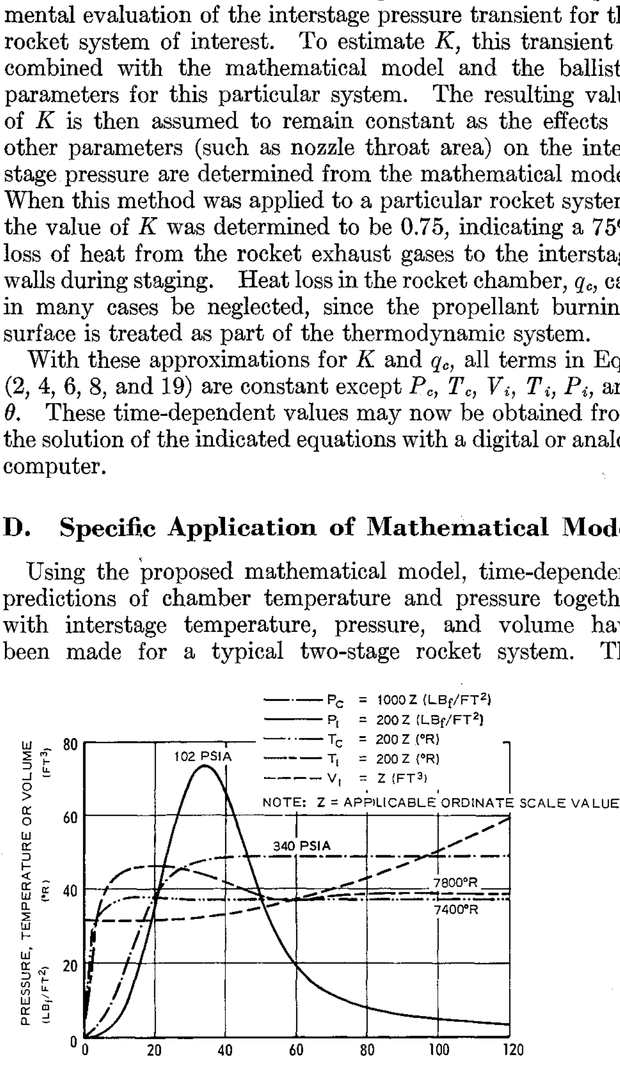

Fig. 3 Prediction of time dependency of interstage pressure for a typical two-stage rocket system.

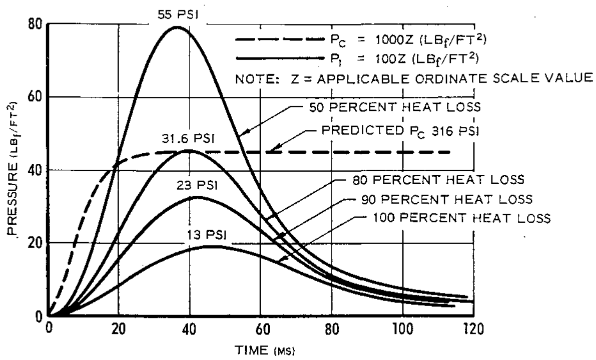

Fig. 4 Effect of heat loss from exhaust gases to interstage casing on predicted interstage pressure.

ballistic parameters shown in Table 1 were substituted into the mathematical model and programmed on a Uni-verse Analog Computer at Hercules Powder Company. For the determination of results, 75% heat loss to the interstage walls and 20% heat loss in the rocket chamber were assumed. As illustrated in Fig. 2, the maximum predicted interstage pressure of 34 psi occurred about 45 msec after upper-stage ignition. For comparison, the interstage pressure prediction for the adiabatic case is shown in Fig. 3. Here, the predicted maximum interstage pressure is 102 psi after 34 msec. Figure 4 shows various estimates of percent heat loss to the interstage region and the effect of each heat loss on the predicted interstage pressure. Comparisons of predicted and experimentally determined interstage pressures reported in the classified literature[3] demonstrate that the proposed mathematical model yields reliable results.

E. Improvement of the Model

The accuracy of the mathematical model in predicting interstage pressure and separation time can be substantially improved by incorporating experimental chamber pressure transients into the calculation. When an experimental chamber pressure transient is available, then Eq. (4) may be deleted from the mathematical model equations and the pressure transient included in its place. This procedure has been successfully used and has resulted in greatly improved accuracy of predicted separation time.[3]

A major limitation of the model presently results from assumption 11: that the heat loss from the gaseous, chamber-exhaust products to the interstage walls is a constant fraction of the thermal energy of the gases. In reality, this heat loss is time dependent, since both the temperatures of the gases and of the interstage walls change with time. A method has recently been devised which accounts for this time dependence of heat loss to the interstage walls. In developing this method, it was assumed that the heat transfer coefficient representing transfer of heat from the hot gases to the interstage walls is constant. Therefore,

$$q_i = dQ_i/d\theta = h_iS(T_i - T_s) \qquad (23)$$

Also, assuming that no temperature gradient exists for the interstage walls and that no heat loss from the interstage to the surrounding atmosphere occurs, then, by an energy balance,

$$Q_i = C_{Ps}W_s(T_s - T_r) \qquad (24)$$

Rearranging Eqs. (24) and (25) to eliminate T_s gives

$$(dQ_i/d\theta) + (h_iS/C_{Ps}W_s)Q_i = h_iS(T_i - T_r) \qquad (25)$$

Accurate analytical evaluation of the product h_iS would be difficult, at best. However, since this quantity has been assumed constant, its value can be readily determined from one experimental interstage pressure run. Equation (25) can be combined with the model equations previously developed, i.e., Eqs. (2, 4, 6, 8, 14, and 19–22), and solved simultaneously to give estimates of P_c, T_c, P_i, V_i, T_i, and q_i as functions of time. A preliminary comparison of interstage pressure transients predicted using this method with available experimental data has been made. Results indicate that this predicted interstage pressure improves the agreement with the experimental data, especially during the first 30 to 50 msec after ignition of the upper stage.

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